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Freddy Van Oystaeyen

Contents

Morita Duality for Corings over Quasi-Frobenius Rings

L. EL KAOUTIT AND J. GOMEZ-TORRECILLAS Departamento de Álgebra, Universidad de Granada, E-18071 Granada, Spain
e-mail address: kaoutit@fedro.ugr.es, torrecil@ugr.es

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ABSTRACT. Given two corings $\mathfrak{C}$ and $\mathfrak{D}$ projective over their Quasi-Frobenius ground rings, then there is a Morita duality between their categories of comodules $\mathcal{M}^{\mathfrak{C}}$ and ${}^{\mathfrak{D}}\mathcal{M}$ if and only if $\mathfrak{C}$ and $\mathfrak{D}$ are Morita-Takeuchi equivalent and semiperfect.

1. INTRODUCTION

Morita duality for coalgebras over fields is tightly linked to the notion of a semiperfect coalgebra. This was shown in [15, 16], with the help of Lin's fundamental paper [20]. Although the proofs (and hence, the results) from these papers can be transferred to coalgebras over Quasi-Frobenius commutative rings, one soon realizes that this is not the case for corings over Quasi-Frobenius rings. This is, apart from the own difficulties of the use of bimodules, ultimately due to the fact that every coring has two different convolution rings. Here, we show how to overcome these technical difficulties. Our approach is based essentially on three pillars: an appropriate way to extend the dual functors from modules over the ground ring to comodules, some ideas coming from the study of semiperfect coalgebras developed in [20, 15, 16, 25] and [21], and the role played in duality by linearly compact subcategories accordingly to Gómez Pardo and Guil approach [11, 12, 17].

2. THE DUAL FUNCTORS

In this paper, all rings have a unit. The opposite ring of a ring R will be denoted by R^{op}. All undefined notions can be found in [2] and [22]. The category of all right (resp. left) modules over a ring R is denoted by $\mathcal{M}_R$ (resp. ${}_R\mathcal{M}$). A *coring* over a ring A (an A–coring, for short) is an A–bimodule $\mathfrak{C}$ together with two A–bimodule homomorphisms $\Delta : \mathfrak{C} \to \mathfrak{C} \otimes_A \mathfrak{C}$ (comultiplication) and $\epsilon : \mathfrak{C} \to A$ (counit) sastisfying the usual coassociative and counitary properties [23]. A right $\mathfrak{C}$–comodule is a right A–module M with a homomorphism of A–modules ρ_M :

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$M \to M \otimes_A \mathfrak{C}$ verifying natural axioms. These are the objects of the additive category $\mathcal{M}^{\mathfrak{C}}$ of all right $\mathfrak{C}$–comodules, whose morphisms are defined in a natural way. For right $\mathfrak{C}$–comodules M and N, the notation $\mathrm{Hom}_{\mathfrak{C}}(M,N)$ stands for the abelian group of all morphisms in $\mathcal{M}^{\mathfrak{C}}$ from M to N. The category $\mathcal{M}^{\mathfrak{C}}$ is easily shown to have arbitrary coproducts and co-kernels, which can be already computed in the category $\mathcal{M}_A$ of all right modules over A. If ${}_A\mathfrak{C}$, the left A–module underlying $\mathfrak{C}$, is flat, then $\mathcal{M}^{\mathfrak{C}}$ is an abelian category. The converse is not true (see [8, Example 1.1]). The category ${}^{\mathfrak{C}}\mathcal{M}$ of left $\mathfrak{C}$–comodules is understood analogously.
Given any right A–module M, we refer to $M^* = \mathrm{Hom}_A(M, A_A)$ as its *dual* left A–module, where the left A–action is given, as usual, by $(af)(m) = af(m)$, for every $a \in A, m \in M, f \in M^*$. This leads to the well known contravariant functor $(-)^* : \mathcal{M}_A \to {}_A\mathcal{M}$ which acts on a morphism $f : M \to N$ in $\mathcal{M}_A$ as $f^*(\alpha) = \alpha \circ f$, for every $\alpha \in M^*$. Of course, there is an analogous contravariant functor ${}^*(-) : {}_A\mathcal{M} \to \mathcal{M}_A$. Our first objective is to extend these dual functors to comodules; some few preliminaries on rational modules are needed.
The coring itself can be seen as a right $\mathfrak{C}$–comodule with coaction Δ. Given a right $\mathfrak{C}$–comodule M we have a natural isomorphism of left A–modules $\mathrm{Hom}_{\mathfrak{C}}(M,\mathfrak{C}) \cong M^*$ that maps a homomorphism f onto $\epsilon \circ f$. For a left $\mathfrak{C}$–comodule M we have an isomorphism of right A–modules $\mathrm{Hom}_{\mathfrak{C}}(M,\mathfrak{C}) \cong {}^*M$. The product (composition) in the ring $\mathrm{End}({}_{\mathfrak{C}}\mathfrak{C})$ is then transferred to the convolution product on ${}^*\mathfrak{C}$, which reads $\sigma\tau = \sigma \circ (\mathfrak{C} \otimes_A \tau) \circ \Delta$ for $\sigma, \tau \in {}^*\mathfrak{C}$ [23]. Analogously, we have an isomorphism of rings $\mathrm{End}(\mathfrak{C}_{\mathfrak{C}})^{\mathrm{op}} \cong \mathfrak{C}^*$, where the convolution product in $\mathfrak{C}^*$ is now given by $\sigma\tau = \tau \circ (\sigma \otimes_A \mathfrak{C}) \circ \Delta$. Every right comodule $\rho_M : M \to M \otimes_A \mathfrak{C}$ leads in this way to a left ${}^*\mathfrak{C}$–module structure on M given by the ring homomorphism ${}^*\mathfrak{C} \to \mathrm{End}(M_{\mathbb{Z}})$ that maps $\sigma \in {}^*\mathfrak{C}$ onto $(M \otimes_A \sigma) \circ \rho_M$. We have in this way a faithful functor $\omega^r : \mathcal{M}^{\mathfrak{C}} \to {}_{{}^*\mathfrak{C}}\mathcal{M}$. This functor becomes an inclusion of $\mathcal{M}^{\mathfrak{C}}$ as a full subcategory of ${}_{{}^*\mathfrak{C}}\mathcal{M}$ when ${}_A\mathfrak{C}$ is locally projective (see [26, 3.5]). Locally projective modules are flat [27, Theorem 2.1]. Such an inclusion can be obtained by recognizing the image of ω^r in ${}_{{}^*\mathfrak{C}}\mathcal{M}$ as the full subcategory of all rational left ${}^*\mathfrak{C}$–modules. This can be done as follows (see [8] or [1]): define the rational submodule $\mathrm{Rat}^l_{\mathfrak{C}}(M)$ of a left ${}^*\mathfrak{C}$–module M as the subset of those elements $m \in M$ for which there are finitely many $(m_i, c_i) \in M \times \mathfrak{C}$ (called a system of rational parameters) such that $\sigma m = \sum_i m_i \sigma(c_i)$ for every $\sigma \in {}^*\mathfrak{C}$. That this construction is mathematically sound is guaranteed here by the locally projective condition on ${}_A\mathfrak{C}$. By [8, Theorem 2.6], ω^r becomes an isomorphism of categories between $\mathcal{M}^{\mathfrak{C}}$ and $\mathrm{Rat}^l_{\mathfrak{C}}({}_{{}^*\mathfrak{C}}\mathcal{M})$, where this last is the full subcategory of ${}_{{}^*\mathfrak{C}}\mathcal{M}$ whose objects are the modules M satisfying $\mathrm{Rat}^l_{\mathfrak{C}}(M) = M$.

PROPOSITION 2.1. *Let $\mathfrak{C}$ be an A–coring.*

a) *The dual M^* of every right $\mathfrak{C}$–comodule M has a structure of right $\mathfrak{C}^*$-module given by*

$$x.\sigma = \sigma \circ (x \otimes_A \mathfrak{C}) \circ \rho_M, \quad \text{for any } x \in M^* \text{ and } \sigma \in \mathfrak{C}^*. \tag{1}$$

In this way, the dual functor $(-)^* : \mathcal{M}_A \to {}_A\mathcal{M}$ extends to a functor $(-)^* : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}_{\mathfrak{C}^*}$ making the diagram (2) commutative, where $U^r : \mathcal{M}_{\mathfrak{C}^*} \to {}_A\mathcal{M}$ is the restriction of scalars functor associated to the canonical ring homomorphism $\rho : A^{\mathrm{op}} \to \mathfrak{C}^*$ which sends $a \in A$ to $\rho(a) : c \mapsto a\epsilon(c)$.

$$\begin{array}{ccc} \mathcal{M}^{\mathfrak{C}} & \xrightarrow{(-)^*} & \mathcal{M}_{\mathfrak{C}^*} \\ {\scriptstyle U_A}\downarrow & & \downarrow{\scriptstyle U^r} \\ \mathcal{M}_A & \xrightarrow{(-)^*} & {}_A\mathcal{M} \end{array} \qquad (2)$$

b) *The dual* $^*N$ *of every left* $\mathfrak{C}$*–comodule* N *has a structure of left* $^*\mathfrak{C}$*-module given by*

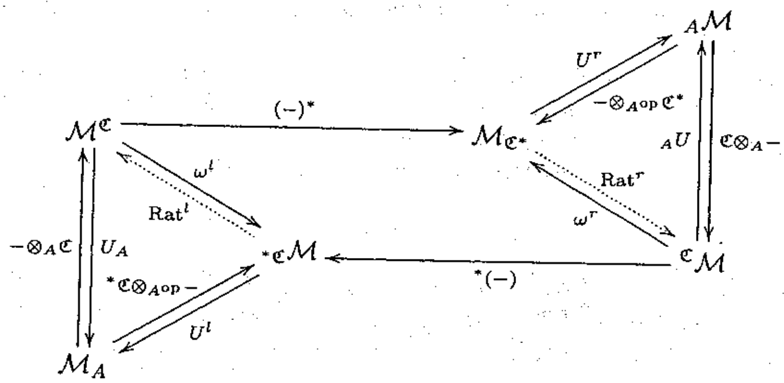
$$\delta.y = \delta \circ (\mathfrak{C} \otimes_A y) \circ \lambda_N, \quad \textit{for any } y \in {}^*N \textit{ and } \delta \in {}^*\mathfrak{C}.$$

In this way, the dual functor $^*(-) : {}_A\mathcal{M} \to \mathcal{M}_A$ *extends to a functor* $^*(-) : {}^{\mathfrak{C}}\mathcal{M} \to {}_{^*\mathfrak{C}}\mathcal{M}$ *making diagram (3) commutative, where* $U^l : {}_{^*\mathfrak{C}}\mathcal{M} \to \mathcal{M}_A$ *is the restriction of scalars functor associated to the canonical ring homomorphism* $\lambda : A^{\mathrm{op}} \to {}^*\mathfrak{C}$ *which sends* $a \in A$ *to* $\lambda(a) : c \mapsto \epsilon(c)a$.

$$\begin{array}{ccc} {}^{\mathfrak{C}}\mathcal{M} & \xrightarrow{^*(-)} & {}_{^*\mathfrak{C}}\mathcal{M} \\ {\scriptstyle {}_AU}\downarrow & & \downarrow{\scriptstyle U^l} \\ {}_A\mathcal{M} & \xrightarrow{^*(-)} & \mathcal{M}_A \end{array} \qquad (3)$$

Proof. We will only show (a), because (b) is similarly proved. For every right $\mathfrak{C}$–comodule M, the abelian group $\mathrm{Hom}_{\mathfrak{C}}(M, \mathfrak{C})$ becomes a left module over $\mathrm{End}(\mathfrak{C}_{\mathfrak{C}})$ in a canonical way. Now, use the ring isomorphism $\mathrm{End}(\mathfrak{C}_{\mathfrak{C}})^{\mathrm{op}} \cong \mathfrak{C}^*$ and the isomorphism $\mathrm{Hom}_{\mathfrak{C}}(M, \mathfrak{C}) \cong M^*$ to transfer this left action to a structure of right $\mathfrak{C}^*$–module over M^*. Some straightforward computations show that this right action is precisely the given in (1). Since the isomorphism $\mathrm{Hom}_{\mathfrak{C}}(-, \mathfrak{C}) \cong (-)^*$ is natural, we have that every morphism f in $\mathcal{M}^{\mathfrak{C}}$ gives a morphism f^* in $\mathcal{M}_{\mathfrak{C}^*}$. Therefore, we have a contravariant functor $(-)^* : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}_{\mathfrak{C}^*}$. The commutativity of the displayed functorial diagram comes from the fact that the ring isomorphism $\mathrm{End}(\mathfrak{C}_{\mathfrak{C}})^{\mathrm{op}} \cong \mathfrak{C}^*$ acts as the identity over A^{op}. □

The following (non-commutative) diagram of functors summarizes the situation, where the dotted arrows make sense under the locally projective condition on the suitable side



The sideways pairs of arrows represent the canonical adjunctions: $-\otimes_A \mathfrak{C}$ is left adjoint to U_A (see [18, Proposition 3.1], [4, Lemma 3.1]), the adjunction between ${}^*\mathfrak{C} \otimes_{A^{\mathrm{op}}} -$ and U^l is the associated to the ring homorphism $A^{\mathrm{op}} \to {}^*\mathfrak{C}$, and the embedding functor $\omega^l : \mathcal{M}^{\mathfrak{C}} = \mathrm{Rat}^l({}_{^*\mathfrak{C}}\mathcal{M}) \to {}_{^*\mathfrak{C}}\mathcal{M}$ is, as usual, left adjoint to the left exact preradical $\mathrm{Rat}^l : {}_{^*\mathfrak{C}}\mathcal{M} \to {}_{^*\mathfrak{C}}\mathcal{M}$. Using these adjunctions, one can easily deduce the following.

Lemma 2.2. *Assume ${}_A\mathfrak{C}$ to be locally projective, and let M be a right $\mathfrak{C}$–comodule. Then*

a) *$M_{\mathfrak{C}}$ is finitely generated (in $\mathcal{M}^{\mathfrak{C}}$) if and only if M_A is finitely generated if and only if ${}_{^*\mathfrak{C}}M$ is finitely generated.*
b) *If $M_{\mathfrak{C}}$ is finitely presented (in $\mathcal{M}^{\mathfrak{C}}$), then M_A is finitely presented.*

Proof. For the proof of (a), use that an object X in a Grothendieck category $\mathcal{A}$ is finitely generated if and only if $\mathrm{Hom}_{\mathcal{A}}(X,-)$ preserves direct unions. If, in addition, $\mathcal{A}$ is locally finitely generated, then X is finitely presented if and only if $\mathrm{Hom}_{\mathcal{A}}(X,-)$ preserves direct limits [22, Chapter V, Prop. 3.4]. Since the cyclic rational left ${}^*\mathfrak{C}$–modules form a generating set for the category $\mathcal{M}^{\mathfrak{C}}$, we easily deduce (b). □

There is a natural transformation $\Phi : 1 \to {}^*(-) \circ (-)^*$ defined, for every right A–module M, by the evaluation map

$$\Phi_M : M \to {}^*(M^*),\ m \mapsto [x \mapsto x(m)]$$

Analogously, there is a natural transformation $\Phi' : 1 \to (-)^* \circ {}^*(-)$ for left A–modules. Assume that ${}_A\mathfrak{C}$ and $\mathfrak{C}_A$ are locally projective. For every right $\mathfrak{C}$–comodule M, let $j : \mathrm{Rat}^r_{\mathfrak{C}}(M^*) \to M^*$ denote the inclusion map and define $\sigma_M : M \to {}^*(\mathrm{Rat}^r_{\mathfrak{C}}(M^*))$ as ${}^*j \circ \Phi_M$. Similarly, we can define $\sigma'_N : N \to (\mathrm{Rat}^l_{\mathfrak{C}}({}^*N))^*$ for every left $\mathfrak{C}$–comodule N.

Proposition 2.3. *Let $\mathfrak{C}$ be an A-coring such that ${}_A\mathfrak{C}$ and $\mathfrak{C}_A$ are locally projective modules. Let M (resp. N) be a right (resp. left) $\mathfrak{C}$–comodule.*

a) *σ_M is left ${}^*\mathfrak{C}$–linear and σ'_N is right $\mathfrak{C}^*$–linear.*
b) *If M_A (resp. ${}_AN$) is finitely presented, then the dual M^* (resp. ${}^*N$) is a left (resp. right) $\mathfrak{C}$–comodule and Φ_M is left ${}^*\mathfrak{C}$–linear (resp. Φ'_N is right $\mathfrak{C}^*$–linear).*
c) *Assume that A is right and left self-injective, then Φ_M (resp. Φ'_N) is a natural ${}^*\mathfrak{C}$–isomorphism (resp. $\mathfrak{C}^*$–isomorphism) for every $M_{\mathfrak{C}}$ (resp. ${}_{\mathfrak{C}}N$) such that M_A (resp. ${}_AN$) is finitely presented.*
d) *If A is right and left Noetherian, then we have a pair of functors*

$$(-)^* : \mathcal{M}_f^{\mathfrak{C}} \leftrightarrows {}_f^{\mathfrak{C}}\mathcal{M} : {}^*(-),$$

where $\mathcal{M}_f^{\mathfrak{C}}$ (resp. ${}_f^{\mathfrak{C}}\mathcal{M}$) denotes the category of all finitely generated right (resp. left) $\mathfrak{C}$–comodules. Moreover, we have natural transformations $\Phi : 1_{\mathcal{M}_f^{\mathfrak{C}}} \to {}^*(-) \circ (-)^*$ and $\Phi' : 1_{{}_f^{\mathfrak{C}}\mathcal{M}} \to (-)^* \circ {}^*(-)$.

Proof. (*a*) A straightforward computation shows that $\mathfrak{C}$ is a ${}^*\mathfrak{C}$-$\mathfrak{C}^*$–bimodule. Consider the commutative diagram

$$\begin{array}{ccc}
M & \xrightarrow{u_M} & \mathrm{Hom}_{\mathfrak{C}^*}(\mathrm{Hom}_{{}^*\mathfrak{C}}(M,\mathfrak{C}),\mathfrak{C}) \\
\Big\downarrow & & \Big\downarrow \cong \\
 & & \mathrm{Hom}_{\mathfrak{C}^*}(M^*,\mathfrak{C}) \\
\Big\downarrow \Phi_M & & \Big\downarrow \mathrm{Hom}_{\mathfrak{C}^*}(j,\mathfrak{C}) \\
 & & \mathrm{Hom}_{\mathfrak{C}^*}(\mathrm{Rat}^r_{\mathfrak{C}}(M^*),\mathfrak{C}) \\
\Big\downarrow & & \Big\downarrow \cong \\
{}^*(M^*) & \xrightarrow{{}^*j} & {}^*(\mathrm{Rat}^r_{\mathfrak{C}}(M^*)),
\end{array}$$

where u_M is the canonical evaluation map, which is left ${}^*\mathfrak{C}$–linear. The left ${}^*\mathfrak{C}$–linear isomorphisms in the diagram are obtained from the isomorphisms $M^* \cong \mathrm{Hom}_{\mathfrak{C}}(M,\mathfrak{C}) = \mathrm{Hom}_{{}^*\mathfrak{C}}(M,\mathfrak{C})$ and $\mathrm{Hom}_{\mathfrak{C}^*}(\mathrm{Rat}^r_{\mathfrak{C}}(M^*),\mathfrak{C}) = \mathrm{Hom}_{\mathfrak{C}}(\mathrm{Rat}^r_{\mathfrak{C}}(M^*),\mathfrak{C}) \cong {}^*(\mathrm{Rat}^r_{\mathfrak{C}}(M^*))$. Thus, $\sigma_M = {}^*j \circ \Phi_M$ becomes a composition of left ${}^*\mathfrak{C}$–linear maps. The right $\mathfrak{C}^*$–linearity of σ'_N is similarly showed.

(*b*) If M_A is finitely presented, as $\mathfrak{C}_A$ is flat, then we have a natural isomorphism

$$\begin{array}{rcl}
\eta_M : \mathfrak{C} \otimes_A \mathrm{Hom}_A(M_A, A) & \longrightarrow & \mathrm{Hom}_A(M_A, \mathfrak{C}) \\
c \otimes_A x & \longmapsto & [m \mapsto cx(m)].
\end{array}$$

Fix $x \in M^*$, and consider an arbitrary $\sigma \in \mathfrak{C}^*$, so $(x \otimes_A \mathfrak{C}) \circ \rho_M \in \mathrm{Hom}_A(M_A, \mathfrak{C})$. Therefore, there exists a finite subset $\{(c_i, x_i)\}$ of $\mathfrak{C} \times M^*$ such that $\eta_M(\sum_i c_i \otimes_A x_i)(m) = (x \otimes_A \mathfrak{C}) \circ \rho_M(m) = \sum_i c_i x_i(m)$, for any $m \in M$. Hence $x.\sigma = \sigma \circ (x \otimes_A \mathfrak{C}) \circ \rho_M = \sum_i \sigma \circ (c_i \otimes_A x_i)$, thus $x.\sigma(m) = \sum_i \sigma(c_i) x_i(m)$, for any $m \in M$; that is $x.\sigma = \sum_i \sigma(c_i).x_i$. This implies that $\{(c_i, x_i)\}$ is a set of rational parameters for x. By [8, Theorem 2.6], $M^* \in \mathrm{Rat}^r_{\mathfrak{C}}(\mathcal{M}_{\mathfrak{C}^*}) \cong {}^{\mathfrak{C}}\mathcal{M}$.

(*c*) This is a consequence of (*a*) and (*b*) in conjunction with [22, p. 47].

(*d*) Let $M \in \mathcal{M}^{\mathfrak{C}}_f$ then, by Lemma 2.2, M_A is finitely generated and, since A is noetherian, M_A is finitely presented. By (b), M^* is a left $\mathfrak{C}$–comodule which is finitely presented as a left A–module because A is noetherian. Therefore, $M^* \in {}^{\mathfrak{C}}_f\mathcal{M}$, and we have thus defined, with the help of Proposition 2.1, the functor $(-)^* : \mathcal{M}^{\mathfrak{C}}_f \to {}^{\mathfrak{C}}_f\mathcal{M}$. The functor ${}^*(-)$ is similarly obtained. □

REMARK 2.4. If $M_{\mathfrak{C}}$ is a comodule such that M_A is finitely presented and $\mathfrak{C}_A$ is flat, then the composite map $M^* \cong \mathrm{Hom}_{\mathfrak{C}}(M,\mathfrak{C}) \subseteq \mathrm{Hom}_A(M,\mathfrak{C}) \cong \mathfrak{C} \otimes_A M^*$ gives a structure of left $\mathfrak{C}$–comodule over M^* as in the case of coalgebras [6], [25, 5.4]. This structure coincides with the given in Proposition 2.3.(b).

3. SEMIPERFECT CORINGS OVER QUASI-FROBENIUS RINGS

A coring is said to be *right semiperfect* if its category of right comodules is a Grothendieck category, and each finitely generated right comodule has a projective cover. A well understood class of semiperfect corings is the given by the cosemisimple corings [8], [7], [14].

Here we consider semiperfect corings as a generalization of semiperfect coalgebras over a field. Most of the results on semiperfect coalgebras can be extended to the coring case, whenever the ground ring is assumed to be Quasi-Frobenius. However, although the proofs for coalgebras over fields can be directly transferred to the case of coalgebras over commutative QF rings, we realized that this is not the case for corings over QF rings. Nevertheless, the approach given in [25] and [21] to coalgebras over commutative QF rings helped us to overcome some technical difficulties in this section.

We use the notation $\mathcal{A}_f$ to designate the full subcategory of a Grothendieck category $\mathcal{A}$ whose objects are the finitely generated ones. If $F : \mathcal{A} \to \mathcal{B}$ is a functor between Grothendieck categories, then the notation $F_f : \mathcal{A}_f \to \mathcal{B}_f$ refers to the restriction functor, whenever F preserves finitely generated objects. Some variations of this notation (like $\mathcal{A}^f, {}^f\!\mathcal{A}, {}_f\mathcal{A}$) are allowed for aesthetical reasons. Thus, the category of all finitely generated right modules over a ring A is $\mathcal{M}_A^f$, while the category of finitely generated left A–modules is ${}_A^f\mathcal{M}$. Recall that a ring A is said to be Quasi-Frobenius (QF, for short) if the functors $(-)^* : \mathcal{M}_A^f \leftrightarrows {}_A^f\mathcal{M} : {}^*(-)$ give a contravariant equivalence of categories. Quasi-Frobenius rings are characterized in several ways, for example, they are just the artinian selfinjective rings. In particular, every flat module is projective over a QF ring. Given a comodule M over a coring, we write $E(M)$ to denote its injective envelope in and $\mathrm{Soc}(M)$ for its socle. Notice that $E(M)$ do exist whenever the corresponding category of comodules is a Grothendieck category.
Recall from [9, page 356] that a Grothendieck category $\mathcal{A}$ is said to be *locally finite* if $\mathcal{A}$ has a generating set consisting of objects of finite length.

Proposition 3.1. *Let A be a QF ring, and $\mathfrak{C}$ be an A–coring such that ${}_A\mathfrak{C}$ is a projective module. Then*

a) *$\mathfrak{C}_\mathfrak{C}$ is an injective comodule and every right $\mathfrak{C}$–comodule embeds in a coproduct of copies of $\mathfrak{C}_\mathfrak{C}$.*
b) *$\mathcal{M}^\mathfrak{C}$ is a locally finite category, in particular $\mathfrak{C}_\mathfrak{C} = \oplus_{\omega\in\Omega} E(S_\omega)^{(n_\omega)}$, where $\{S_\omega\}_{\omega\in\Omega}$ is the set of all representatives of simple right $\mathfrak{C}$–comodules, and the n_ω's are cardinal numbers.*

Proof. (a) The ring A is right selfinjective and all right modules embed in free modules (by the Faith-Walker characterization of QF rings). These properties are transferred to $\mathfrak{C}$ because the exact functor $- \otimes_A \mathfrak{C} : \mathcal{M}_A \to \mathcal{M}^\mathfrak{C}$ preserves direct sums and has an exact left adjoint.
(b) This is a consequence of (*a*) and Lemma 2.2. □

The following consequence of Proposition 2.3 is crucial for the development of the theory.

Theorem 3.2. *Let $\mathfrak{C}$ be a coring over a QF ring A such that ${}_A\mathfrak{C}$ and $\mathfrak{C}_A$ are projective modules. Then we have a contravariant equivalence of categories*

$$(-)^* : \mathcal{M}_f^\mathfrak{C} \leftrightarrows {}_f^\mathfrak{C}\mathcal{M} : {}^*(-)$$

The following proposition illustrates the use of Theorem 3.2 in conjunction with the locally finite property for comodules given in Proposition 3.1.

PROPOSITION 3.3. *Let $\mathfrak{C}$ be a coring over a QF ring A such that ${}_A\mathfrak{C}$ and $\mathfrak{C}_A$ are projective modules. Consider $M \in \mathcal{M}_f^{\mathfrak{C}}$, a finitely generated comodule. Then*

a) *If $M_{\mathfrak{C}}$ is an injective comodule, then $M^*_{\mathfrak{C}^*}$ is a projective module.*
b) *$M_{\mathfrak{C}}$ is projective if and only if ${}_{\mathfrak{C}}M^*$ is injective.*
c) *$M_{\mathfrak{C}}$ is injective if and only if ${}_{\mathfrak{C}}M^*$ is projective.*
d) *$M_{\mathfrak{C}}$ is a projective comodule if and only if ${}_{{}^*\mathfrak{C}}M$ is a projective module.*

Proof. (*a*) We use an argument taken from the proof of [6, Proposition 4]. If M is an injective right $\mathfrak{C}$–comodule then, by Proposition 3.1, there is a splitting monomorphism of right comodules $M \hookrightarrow \mathfrak{C}^{(n)}$, for some finite index n. By applying the functor $(-)^* : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}_{\mathfrak{C}^*}$ given in Proposition 2.1, we obtain a splitting epimorphism of right $\mathfrak{C}^*$–modules ${\mathfrak{C}^*}^{(n)} \twoheadrightarrow M^*$ which, of course, implies that M^* is a projective right $\mathfrak{C}^*$–module.
(*b*) If $M_{\mathfrak{C}}$ is a projective comodule, then, by Theorem 3.2, M^* is L–injective for every finitely generated left $\mathfrak{C}$–comodule L. Consequently, by [2, Proposition 16.13.(2)], M^* is $(\oplus L_i)$–injective for every family $\{L_i\}_i$ of finitely generated left $\mathfrak{C}$–comodules. Therefore, by [2, 16.13.(1)], and Proposition 3.1.(b), M^* is an injective left $\mathfrak{C}$–comodule. Conversely, suppose that M^* is an injective left comodule. By the version of part (*a*) for left comodules, it follows that $M \cong {}^*(M^*)$ is a projective left ${}^*\mathfrak{C}$–module, and, by the isomorphism of categories $\mathrm{Rat}^l_{\mathfrak{C}}({}_{{}^*\mathfrak{C}}\mathcal{M}) \cong \mathcal{M}^{\mathfrak{C}}$, a projective right $\mathfrak{C}$–comodule.
(*c*) Write $N = {}_{\mathfrak{C}}M^*$, and apply the version of part (*b*) to left comodules to obtain that ${}_{\mathfrak{C}}N$ is projective if and only if ${}^*N_{\mathfrak{C}}$ is injective. Finally, apply the duality stated in Theorem 3.2.
(*d*) The proof of (*b*) runs here to prove that if $M_{\mathfrak{C}}$ is projective then ${}_{{}^*\mathfrak{C}}M$ is projective. □

A *projective cover* of an object X of a Grothendieck category $\mathcal{A}$ is an epimorphism $p : P \to X$ in $\mathcal{A}$ such that P is a projective object and $Ker(p)$ is a superfluous (or small) subobject of P (see, e.g. [22, Chapter V, page 120]).

PROPOSITION 3.4. *Let $\mathfrak{C}$ be a coring over a QF ring A such that ${}_A\mathfrak{C}$ and $\mathfrak{C}_A$ are projective modules. A finitely generated right $\mathfrak{C}$–comodule M has a projective cover in $\mathcal{M}^{\mathfrak{C}}$ if and only if ${}_AE({}_{\mathfrak{C}}M^*)$ is finitely generated.*

Proof. Assume that $P \twoheadrightarrow M$ is a projective cover of M in $\mathcal{M}^{\mathfrak{C}}$. By Proposition 3.3.(d), $P \twoheadrightarrow M$ is a projective cover of M in ${}_{{}^*\mathfrak{C}}\mathcal{M}$, as the lattices of subobjects of ${}_{{}^*\mathfrak{C}}P$ and $P_{\mathfrak{C}}$ are equal. Then P is a finitely generated left ${}^*\mathfrak{C}$–module and, by Corollary 2.2, it is a finitely generated right A–module. By Theorem 3.2 and Proposition 3.3.(b) we have that the dual map $M^* \hookrightarrow P^*$ gives an injective hull of M^* in ${}^{\mathfrak{C}}\mathcal{M}$. Thus, we can write $E(M^*) = P^*$, which is finitely generated as a left A–module.
If ${}_AE(M^*)$ is finitely generated, then, by Theorem 3.2 and Proposition 3.3.(d), the dual map ${}^*(E(M^*)) \twoheadrightarrow {}^*(M^*) \cong M$ is a projective cover. □

The following theorem was first proved for coalgebras by Lin in [20].

THEOREM 3.5. *Let A, $\mathfrak{C}$ be as in Proposition 3.4. The following statements are equivalent.*

(i) *Every simple right $\mathfrak{C}$–comodule has a projective cover;*
(ii) *every finitely generated right $\mathfrak{C}$–comodule has a projective cover (i.e. $\mathfrak{C}$ is right semiperfect);*
(iii) *${}_AE(N)$ is finitely generated for every finitely generated left $\mathfrak{C}$–comodule N;*
(iv) *${}_AE(S)$ is finitely generated for every simple left $\mathfrak{C}$–comodule S.*

Proof. *(i)* $\Rightarrow$ *(iv)* Let S be a simple left $\mathfrak{C}$–comodule. By Theorem 3.2 we know that ${}^*S$ is a simple right $\mathfrak{C}$–comodule. By hypothesis, ${}^*S$ has a projective cover which implies, by Proposition 3.4, that ${}_AE(({}^*S)^*)$ is finitely generated. Thus, ${}_AE(S)$ is finitely generated, as $S \cong ({}^*S)^*$.
(iv) $\Rightarrow$ *(iii)* By Proposition 3.1, $E(N) = E(\mathrm{Soc}(N))$ and, since N is finitely generated, $\mathrm{Soc}(N) = S_1 \oplus \cdots \oplus S_r$, for some simple left comodules $S_1, \ldots, S_r$. Therefore, $E(N) = E(S_1) \oplus \cdots \oplus E(S_r)$ and, in particular, it is finitely generated as a left A–module.
(iii) $\Rightarrow$ *(ii)* If M is a finitely generated right $\mathfrak{C}$–comodule, then M^* is a finitely generated left $\mathfrak{C}$–comodule which, by hypothesis, has a finitely generated injective hull. By Proposition 3.4, M possesses a projective cover.
(ii) $\Rightarrow$ *(i)* This is obvious. □

The next step is to characterize right semiperfect corings in terms of density. The notion of density we will use is analogue to the usually considered for coalgebras over fields. Let M be a right A–module over a QF ring A, and consider the pairing

$$\langle -, - \rangle : M \times M^* \to A \qquad (\langle m, \varphi \rangle = \varphi(m)).$$

For every right A–submodule U of M, define $U^\perp = \{\varphi \in M^* : \langle U, \varphi \rangle = 0\}$, which is a left A–submodule of M^*. In fact, the mapping $(M/U)^* \to U^\perp$ that sends $\alpha \in (M/U)^*$ to $\alpha \circ \pi$, where $\pi : M \to M/U$ is the canonical projection, is an isomorphism of left A–modules. An analogous definition of $V^\perp$ can be made for left A–submodules V of M^*. Such a submodule V of M^* is said to be *dense* if $V^\perp = 0$. The fact that A_A is an injective cogenerator for $\mathcal{M}_A$ guarantees that V is dense in M^* if and only if $V^{\perp\perp} = M^*$.

The version for coalgebras of the following Lemma 3.6 is [20, Lemma 7]. The first statement of Proposition 3.7 was proved for coalgebras in [25, 6.1], while the second appears in [21, Lemma 1.8].

LEMMA 3.6. *Let $\mathfrak{C}$ be an A–coring over a QF ring such that $\mathfrak{C}_A$ is projective. Let $f : M \hookrightarrow N$ be a monomorphism of right $\mathfrak{C}$–comodules. If $\mathrm{Rat}^r_{\mathfrak{C}}(N^*)$ is dense in N^*, then $\mathrm{Rat}^r_{\mathfrak{C}}(M^*)$ is dense in M^*.*

Proof. This is a consequence of the fact that $f^*(\mathrm{Rat}^r_{\mathfrak{C}}(N^*)) \subseteq \mathrm{Rat}^r_{\mathfrak{C}}(M^*)$. □

PROPOSITION 3.7. *Let $\mathfrak{C}$ be a coring over a QF ring A such that ${}_A\mathfrak{C}$ and $\mathfrak{C}_A$ are projectives. Then*

(a) $\mathrm{Soc}(\mathfrak{C}_{\mathfrak{C}}) = \mathrm{Soc}({}_{^*\mathfrak{C}}\mathfrak{C})$ *is essential in $\mathfrak{C}_{\mathfrak{C}}$, and the Jacobson radical of $\mathfrak{C}^*$ is*

$$\mathrm{Jac}(\mathfrak{C}^*) = (\mathrm{Soc}({}_{^*\mathfrak{C}}\mathfrak{C}))^\perp$$

(b) *If S is a right simple comodule, then $E(S)^*$ is a cyclic local right $\mathfrak{C}^*$–module with $S^\perp$ the unique maximal right $\mathfrak{C}^*$-submodule.*

Proof. (*a*) That $\mathrm{Soc}(\mathfrak{C}_\mathfrak{C}) = \mathrm{Soc}({}_{^*\mathfrak{C}}\mathfrak{C})$ is essential in $\mathfrak{C}_\mathfrak{C}$ follows from Proposition 3.1. By Proposition 3.1.(a), ${}_{^*\mathfrak{C}}\mathfrak{C}$ is injective in $\mathcal{M}^\mathfrak{C}$. This implies that the Jacboson radical of $\mathrm{End}(\mathfrak{C}_\mathfrak{C})$ consists of those $f : \mathfrak{C} \to \mathfrak{C}$ with essential kernel, which, in the present case, is equivalent to $f(\mathrm{Soc}(\mathfrak{C}_\mathfrak{C})) = 0$. The ring isomorphism $\mathfrak{C}^* \cong \mathrm{End}(\mathfrak{C}_\mathfrak{C})^{\mathrm{op}}$ gives the description of $\mathrm{Jac}(\mathfrak{C}^*)$.
(*b*) Consider $S_\mathfrak{C}$ a simple comodule, so we have the following commutative diagram with exact rows

$$\begin{array}{ccccc} S & \hookrightarrow & E(S) & \twoheadrightarrow & E(S)/S \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Soc}({}_{^*\mathfrak{C}}\mathfrak{C}) & \hookrightarrow & \mathfrak{C} & \twoheadrightarrow & \mathfrak{C}/\mathrm{Soc}({}_{^*\mathfrak{C}}\mathfrak{C}) \end{array} .$$

Dualising we get

$$\begin{array}{ccccc} (\mathrm{Soc}({}_{^*\mathfrak{C}}\mathfrak{C}))^\perp \cong (\mathfrak{C}/\mathrm{Soc}({}_{^*\mathfrak{C}}\mathfrak{C}))^* & \hookrightarrow & \mathfrak{C}^* & \twoheadrightarrow & \mathrm{Soc}({}_{^*\mathfrak{C}}\mathfrak{C})^* \\ \downarrow & & \downarrow & & \downarrow \\ S^\perp \cong (E(S)/S)^* & \hookrightarrow & E(S)^* & \twoheadrightarrow & S^* \end{array},$$

a commutative diagram in $\mathcal{M}_{\mathfrak{C}^*}$. Clearly, $E(S)^*$ is cyclic, and the exactness of the diagram easily gives $S^\perp = E(S)^*\mathrm{Jac}(\mathfrak{C}^*)$. Therefore, $S^\perp$ is superfluous in $E(S)^*$, whence $S^\perp \subseteq \mathrm{Jac}(E(S)^*)$ (the radical of $E(S)^*_{\mathfrak{C}^*}$). By Theorem 3.2, S^* is simple which implies, in view of the second exact row of our diagram, that $S^\perp$ is a maximal right $\mathfrak{C}^*$–submodule of $E(S)^*$, whence $\mathrm{Jac}(E(S)^*) = S^\perp$. □

We are ready to prove the main result in this section. It generalizes, in conjunction with Theorem 4.2 in the next section, [20, Theorem 10].

THEOREM 3.8. *Let $\mathfrak{C}$ be a coring over a QF ring A such that ${}_A\mathfrak{C}$ and $\mathfrak{C}_A$ are projective modules. The following statements are equivalent.*

(i) *The coring $\mathfrak{C}$ is right semiperfect;*
(ii) $\mathrm{Rat}^l_\mathfrak{C}({}_{^*\mathfrak{C}}{}^*\mathfrak{C})$ *is dense in* ${}^*\mathfrak{C}$;
(iii) $\mathrm{Rat}^l_\mathfrak{C}({}^*E(S))$ *is dense in* ${}^*E(S)$ *for each simple left $\mathfrak{C}$-comodule S.*

Proof. (*i*) ⇒ (*iii*) If S is a simple left $\mathfrak{C}$–comodule, then, by Theorem 3.5, $E(S)$ is a finitely generated left A–module. By Proposition 2.3, ${}^*E(S)$ is a right $\mathfrak{C}$–comodule. Thus, $\mathrm{Rat}^l_\mathfrak{C}({}^*E(S)) = {}^*E(S)$ and, obviously, $\mathrm{Rat}^l_\mathfrak{C}({}^*E(S))$ is dense in ${}^*E(S)$.
(*iii*) ⇒ (*ii*) By Proposition 3.1, ${}_\mathfrak{C}\mathfrak{C} = \bigoplus_{i\in I} E(S_i)$ for a suitable set S_i of simple left $\mathfrak{C}$–comodules. Clearly, $\bigoplus_{i\in I} {}^*E(S_i)$ is dense in $\prod_{i\in I} {}^*E(S_i) = {}^*\mathfrak{C}$. Therefore $\bigoplus_{i\in I} \mathrm{Rat}^l_\mathfrak{C}({}^*E(S_i))$ is dense in ${}^*\mathfrak{C}$, and so is $\mathrm{Rat}^l_\mathfrak{C}({}_{^*\mathfrak{C}}{}^*\mathfrak{C})$.
(*ii*) ⇒ (*iii*) This is a consequence of (the symmetric version of) Lemma 3.6, as every left comodule of the form $E(S)$ for a simple S embeds in $\mathfrak{C}$.
(*iii*) ⇒ (*i*) Let S be a left $\mathfrak{C}$–comodule, and consider its injective envelope $E(S) \in {}^\mathfrak{C}\mathcal{M}$. By the version of Proposition 3.7.(b) for left comodules, ${}^*E(S)$ is a cyclic local left ${}^*\mathfrak{C}$–module with maximal submodule $S^\perp$. Since $\mathrm{Rat}^l_\mathfrak{C}({}^*E(S))$ is dense in ${}^*E(S)$, we get that $S^\perp \subsetneq S^\perp + \mathrm{Rat}^l_\mathfrak{C}({}^*E(S))$; hence $S^\perp + \mathrm{Rat}^l_\mathfrak{C}({}^*E(S)) = {}^*E(S)$, as

$S^{\perp}$ is maximal. Finally, $S^{\perp}$ is superfluous in ${}^{*}E(S)$, whence $\mathrm{Rat}^{l}_{\mathfrak{C}}({}^{*}E(S)) = {}^{*}E(S)$. Therefore, ${}^{*}E(S)$ is rational and finitely generated ${}^{*}\mathfrak{C}$–module. By Theorem 3.2, $E(S) \cong ({}^{*}E(S))^{*}$ is finitely generated (as an A–module, if desired). □

4. DUALITY FOR SEMIPERFECT CORINGS OVER QF RINGS

This section contains the extension of the duality theory developed in [15, 16] for semiperfect coalgebras over fields to semiperfect corings over QF ground rings. Thus, we assume in this section that the corings are over QF rings.

A *duality* is a contravariant equivalence between two categories. Theorem 3.2 says that the functors $(-)^{*}$ and ${}^{*}(-)$ give a duality between $\mathcal{M}^{\mathfrak{C}}_{f}$ and ${}^{\mathfrak{C}}_{f}\mathcal{M}$. This notion of duality is too restrictive even from the point of view of module theory, where the concept of a Morita duality has been proved to be fundamental. In what follows, let us recall a generalization of Morita duality to Grothendieck categories due to R. R. Colby and K. R. Fuller [5]. Consider contravariant functors between Grothendieck categories

$$H : \mathcal{A} \leftrightarrows \mathcal{A}' : H',$$

together with natural transformations $\tau : 1_{\mathcal{A}} \to H' \circ H$ and $\tau' : 1_{\mathcal{A}'} \to H \circ H'$, satisfying the condition $H(\tau_X) \circ \tau'_{H(X)} = 1_{H(X)}$ and $H'(\tau'_{X'}) \circ \tau_{H'(X')} = 1_{H'(X')}$ for $X \in \mathcal{A}$ and $X' \in \mathcal{A}'$. This situation is called a *right adjoint pair*. Moreover, any pair of natural transformations τ, τ' satisfying these conditions determine a natural isomorphism

$$\begin{array}{rcl} \eta_{X,X'} : \mathrm{Hom}_{\mathcal{A}}(X, H'(X')) & \longrightarrow & \mathrm{Hom}_{\mathcal{A}'}(X', H(X)) . \\ \alpha & \longmapsto & H(\alpha) \circ \tau'_{X'} \\ H'(\beta) \circ \tau_X & \longleftarrow\!\shortmid & \beta \end{array}$$

Conversely, given any natural isomorphism

$$\eta_{X,X'} : \mathrm{Hom}_{\mathcal{A}}(X, H'(X')) \to \mathrm{Hom}_{\mathcal{A}'}(X', H(X)),$$

then $\tau_X = \eta^{-1}_{X,H(X)}(1_{H(X)})$ and $\tau'_{X'} = \eta_{H'(X'),X'}(1_{H'(X')})$ satisfy the above conditions. We call an object X of $\mathcal{A}$ (resp. X' of $\mathcal{A}'$) *reflexive* in case τ_X (resp. $\tau'_{X'}$) is an isomorphism. If we denote by $\mathcal{A}_0$ and $\mathcal{A}'_0$ the full subcategories of $\mathcal{A}$ and $\mathcal{A}'$ of the reflexive objects, then H and H' form duality between them. Using the terminology of [11] we will say that the pair of right adjoint functors is a *Colby-Fuller* duality between $\mathcal{A}$ and $\mathcal{A}'$ if and only if the functors H and H' are exact and $\mathcal{A}_0$ and $\mathcal{A}'_0$ are closed under subobjects, quotient objects, and finite direct sums (i.e., they are *finitely closed*) and contain sets of generators for $\mathcal{A}$ and $\mathcal{A}'$ (i.e., they are *generating*).

Now, we return to categories of comodules. Let $\mathfrak{C}$ be a coring over a QF ring A such that ${}_{A}\mathfrak{C}$ and $\mathfrak{C}_{A}$ are projective modules. The duality given in Theorem 3.2 is a Morita duality between $\mathcal{M}^{\mathfrak{C}}$ and ${}^{\mathfrak{C}}\mathcal{M}$ in the sense of [3], as the subcategories $\mathcal{M}^{\mathfrak{C}}_{f}$ and ${}^{\mathfrak{C}}_{f}\mathcal{M}$ are generating. According to [12, Theorem 2.1], the functors $(-)^{*} : \mathcal{M}^{\mathfrak{C}}_{f} \rightleftarrows {}^{\mathfrak{C}}_{f}\mathcal{M} : {}^{*}(-)$ can be extended to a right adjoint pair of contravariant functors $D : \mathcal{M}^{\mathfrak{C}} \rightleftarrows {}^{\mathfrak{C}}\mathcal{M} : D'$ in a unique way. For our purposes we will need the following description of this extension. For every $M \in \mathcal{M}^{\mathfrak{C}}$ we get, by Proposition 2.2, a left

${}^*\mathfrak{C}$–linear map $\sigma_M : M \to {}^*(\mathrm{Rat}^r_{\mathfrak{C}}(M^*))$. Since M is rational, we deduce that the image of σ_M is included in $\mathrm{Rat}^l_{\mathfrak{C}}({}^*(\mathrm{Rat}^r_{\mathfrak{C}}(M^*)))$, so that we have obtained a natural transformation

$$\sigma : 1_{\mathcal{M}^{\mathfrak{C}}} \to \mathrm{Rat}^l_{\mathfrak{C}} \circ {}^*(-) \circ \mathrm{Rat}^r_{\mathfrak{C}} \circ (-)^*.$$

Analogously, we obtain a natural transformation

$$\sigma' : 1_{{}^{\mathfrak{C}}\mathcal{M}} \to \mathrm{Rat}^r_{\mathfrak{C}} \circ (-)^* \circ \mathrm{Rat}^l_{\mathfrak{C}} \circ {}^*(-).$$

The functors

$$\mathrm{Rat}^r_{\mathfrak{C}} \circ (-)^* : \mathcal{M}^{\mathfrak{C}} \rightleftarrows {}^{\mathfrak{C}}\mathcal{M} : \mathrm{Rat}^l_{\mathfrak{C}} \circ {}^*(-) \tag{4}$$

and the natural transformations σ and σ' give the aforementioned right adjoint pair extending the duality $(-)^* : \mathcal{M}^{\mathfrak{C}}_f \rightleftarrows {}^{\mathfrak{C}}_f\mathcal{M} : {}^*(-)$.

In [16], the authors characterize right semiperfect coalgebras by the locally compactness of the full subcategory of finite-dimensional right comodules. We shall extend this characterization to corings over a QF ring. Recall, from [10], that an object X of a Grothendieck category $\mathcal{A}$, is called *linearly compact* when for each inverse system of epimorphisms $\{p_i : X \twoheadrightarrow X_i\}_{i \in I}$ in $\mathcal{A}$, the induced morphism $\varprojlim p_i : X \to \varprojlim X_i$ is also an epimorphism. A full subcategory $\mathcal{S}$ of $\mathcal{A}$ is said to be *linearly compact* if for every inverse system $\{S_i \to A_i\}$ of epimorphisms, with $S_i \in \mathcal{S}$, the projective limit $\varprojlim S_i \to \varprojlim A_i$ is an epimorphism. Note that if $\mathcal{S}$ is linearly compact, then each of its objects is also linearly compact; but the converse is not true (see [11, Example 4]).

LEMMA 4.1. *Let A, $\mathfrak{C}$ be as in Theorem 3.8. Suppose that $\mathfrak{C}$ is right semiperfect, and let $P \in \mathcal{M}^{\mathfrak{C}}$ be a projective object. Then ${}_{^*\mathfrak{C}}P$ is a projective module.*

Proof. By the locally finite property for right $\mathfrak{C}$–comodules there is an epimorphism of right comodules

$$\bigoplus_{i \in I} M_i \twoheadrightarrow P$$

or, equivalently, of left ${}^*\mathfrak{C}$–modules, where $\{M_i | i \in I\}$ is a family of finitely generated left ${}^*\mathfrak{C}$–modules. Now consider the following epimorphism

$$\bigoplus_{i \in I} P_i \twoheadrightarrow P, \tag{5}$$

where each $P_i \to M_i$ is a projective cover of M_i, and P_i is finitely generated right $\mathfrak{C}$–comodule. Since $P_{\mathfrak{C}}$ is projective, (5) is splitting so $\oplus_{i \in I} P_i = P \oplus P'$, for some $P'_{\mathfrak{C}}$. By Proposition 3.3, each P_i is a projective left ${}^*\mathfrak{C}$–module, thus ${}_{^*\mathfrak{C}}P$ is a projective module. □

The following result extends characterizations of semiperfect coalgebras over fields and commutative QF rings [20, Theorem 10], [15, Theorem 3.3], [16, Theorem 1.5], [25, 6.3] to corings over QF rings. The category $\mathcal{M}^{\mathfrak{C}}$ is said *to have enough projectives* if it has a projective generator.

THEOREM 4.2. *Let $\mathfrak{C}$ be a coring over a QF ring A such that ${}_A\mathfrak{C}$ and $\mathfrak{C}_A$ are projective modules. The following conditions are equivalent*

(i) *$\mathfrak{C}$ is right semiperfect;*
(ii) *the category $\mathcal{M}^{\mathfrak{C}}$ has enough projectives;*
(iii) *$\mathcal{M}^{\mathfrak{C}}_f$ is a linearly compact subcategory of $\mathcal{M}^{\mathfrak{C}}$;*

(iv) *the functor* $\mathrm{Rat}^l_{\mathfrak{C}} \circ {}^*(-) : {}^{\mathfrak{C}}\mathcal{M} \to {}_{{}^*\mathfrak{C}}\mathcal{M}$ *is exact;*
(v) $\sigma'_M : M \to (\mathrm{Rat}^l_{\mathfrak{C}}({}^*M))^*$ *is a monomorphism for every left* $\mathfrak{C}$*–comodule* M*;*
(vi) $\mathrm{Rat}^l_{\mathfrak{C}}({}^*M)$ *is dense in* ${}^*M$ *for every right* $\mathfrak{C}$*–comodule* M*;*
(vii) *the functor* $\mathrm{Rat}^l_{\mathfrak{C}} : {}_{{}^*\mathfrak{C}}\mathcal{M} \to {}_{{}^*\mathfrak{C}}\mathcal{M}$ *is exact.*

Therefore, if $\mathfrak{C}$ *is right semiperfect, then* $\mathrm{Rat}^l_{\mathfrak{C}}({}_{{}^*\mathfrak{C}}\mathcal{M})$ *is a localizing subcategory of* ${}_{{}^*\mathfrak{C}}\mathcal{M}$.

Proof. *(i)* $\Rightarrow$ *(ii)* Since $\mathcal{M}^{\mathfrak{C}}$ is locally finite, then the coproduct of the projective covers of a complete set of simple right $\mathfrak{C}$–comodules is a projective generator.
(ii) $\Rightarrow$ *(iii)* Let U be a projective generator of $\mathcal{M}^{\mathfrak{C}}$, and $T = \mathrm{End}(U_{\mathfrak{C}})$ its endomorphism ring. The exact faithful and full functor $F = \mathrm{Hom}_{\mathfrak{C}}(U, -) : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}_T$ has a left adjoint $G = - \otimes_T U : \mathcal{M}_T \to \mathcal{M}^{\mathfrak{C}}$, let $\eta : 1 \to FG$ be its unity and $\nu : GF \to 1$ be its counity. Let us first prove that F preserves locally compact objects. Given a locally compact comodule M and an inverse system of epimorphisms $\{F(M) \to N_i\}$ in $\mathcal{M}_T$, we get an inverse system of epimorphisms $\{GF(M) \to G(N_i)\}$ in $\mathcal{M}^{\mathfrak{C}}$, and a new inverse system of epimorphisms of modules $\{FGF(M) \to FG(N_i)\}$. Therefore, we have commuting diagrams

$$\begin{array}{ccc} FGF(M) & \longrightarrow & FG(N_i) \\ {\scriptstyle \eta_{F(M)}} \uparrow & & \uparrow {\scriptstyle \eta_{N_i}} \\ F(M) & \longrightarrow & N_i, \end{array}$$

which give a commutative diagram

$$\begin{array}{ccc} FGF(M) & \longrightarrow & \varprojlim FG(N_i) \cong F(\varprojlim G(N_i)) \\ {\scriptstyle \eta_{F(M)}} \uparrow & & \uparrow \\ F(M) & \longrightarrow & \varprojlim N_i \end{array} \qquad (6)$$

Now, ν_M gives an isomorphism $GF(M) \cong M$ and $\eta_{F(M)} = F(\nu_M^{-1})$ is an isomorphism, too. Therefore, the left vertical arrow in (6) is an isomorphism and the top arrow is an epimorphism (since $GF(M) \cong M$ is linearly compact and F is exact). We get thus that $F(M) \to \varprojlim N_i$ is an epimorphism and $F(M)$ is then linearly compact. Let $\mathcal{C}$ be the image under F of $\mathcal{M}^{\mathfrak{C}}_f$, which is a full subcategory of $\mathcal{M}_T$ consisting of linearly compact modules. By [19, Theorem 7.1] (see also [11, Lemma 6]), $\mathcal{C}$ is a linearly compact subcategory of $\mathcal{M}_T$. Now, given any inverse system of epimorphisms $\{M_i \to L_i\}$ with $M_i \in \mathcal{M}^f_{\mathfrak{C}}$ we get an inverse system of epimorphisms of modules $\{F(M_i) \to F(L_i)\}$. By [12, Proposition 3.1] and Theorem 3.2, the M_i's are linearly compact and, hence, the $F(M_i)$'s are in $\mathcal{C}$. We thus get that $F(\varprojlim M_i) \cong \varprojlim F(M_i) \to \varprojlim F(N_i) \cong F(\varprojlim N_i)$ is an epimorphism. Since F is faithful, we deduce that $\varprojlim M_i \to \varprojlim N_i$ is an epimorphism. Therefore, $\mathcal{M}^{\mathfrak{C}}_f$ is a linearly compact subcategory of $\mathcal{M}^{\mathfrak{C}}$.
(iii) $\Rightarrow$ *(iv)* The proof of [16, Theorem 1.5] given for coalgebras runs in the framework of corings.

(iv) $\Rightarrow$ *(v)* Every $m \in M$ is contained in a $\mathfrak{C}$–subcomodule N of M such that ${}_AN$ is finitely presented. We have the following commutative diagram

$$\begin{array}{ccc} N & \xrightarrow{\Phi'_N} & ({}^*N)^* \\ \Big\downarrow{\scriptstyle i} & & \Big\downarrow{\scriptstyle (\mathrm{Rat}^l_{\mathfrak{C}}({}^*i))^*} \\ M & \xrightarrow{\sigma'_M} & (\mathrm{Rat}^l_{\mathfrak{C}}({}^*M))^*, \end{array} \qquad (7)$$

where i is the inclusion map. The injectivity of the map σ'_M is easily deduced from this diagram, as $(\mathrm{Rat}^l_{\mathfrak{C}}({}^*i))^*$ and Φ'_N are monomorphisms.
(v) $\Rightarrow$ *(vi)* It follows because $Ker\sigma'_M = \mathrm{Rat}^l_{\mathfrak{C}}({}^*M)^{\perp}$.
(vi) $\Rightarrow$ *(i)* Apply Theorem 3.8.
(i) $\Rightarrow$ *(vii)* It follows from Lemma 4.1, since $\omega^l : \mathrm{Rat}^l_{\mathfrak{C}}({}_{^*\mathfrak{C}}\mathcal{M}) \to {}_{^*\mathfrak{C}}\mathcal{M}$ is left adjoint to $\mathrm{Rat}^l_{\mathfrak{C}}$.
(vii) $\Rightarrow$ *(iv)* This is clear, as ${}^*(-)$ is an exact functor. □

REMARK 4.3. Examples of semiperfect corings can be constructed as follows: let U_A be a finitely generated and projective module with dual basis $\{e_i, e_i^*\}$ and let $T \subseteq \mathrm{End}(U_A)$ be a semiperfect subring such that ${}_TU$ is faithfully flat. Consider the *comatrix A–coring* $U^* \otimes_T U$ with comultiplication given by $\Delta(\varphi \otimes_T u) = \sum_i \varphi \otimes_T e_i \otimes_A e_i^* \otimes_T u$ and counit given by $\epsilon(\varphi \otimes_T u) = \varphi(u)$, for $\varphi \otimes_T u \in U^* \otimes_T U$ (see [7]). By [7, Theorem 2], the category $\mathcal{M}^{U^* \otimes_T U}$ is equivalent to $\mathcal{M}_T$ and, hence, $U^* \otimes_T U$ is a right semiperfect A–coring with finitely many non isomorphic simple right comodules (no assumption is made here on the ground ring A). In fact, if A is a QF ring, every right semiperfect A–coring $\mathfrak{C}$ with finitely many simples such that ${}_A\mathfrak{C}$ is projective is a comatrix coring as described before. To see this, consider a set $S_1, \ldots, S_n$ of representatives of all simple right $\mathfrak{C}$–comodules. Then a finitely generated projective generator of $\mathcal{M}^{\mathfrak{C}}$ is $U = P_1 \oplus \cdots \oplus P_n$, where P_i is a projective cover of S_i. Let $T = \mathrm{End}(U_{\mathfrak{C}})$, which is a semiperfect subring of $\mathrm{End}(U_A)$. By [7, Theorem 1], U_A is finitely generated projective and ${}_TU$ is faithfully flat. Moreover, there is a canonical isomorphism of A–corings $can : U^* \otimes_T U \cong \mathfrak{C}$ and the category $\mathcal{M}^{\mathfrak{C}}$ is equivalent to $\mathcal{M}_T$.

We are now ready to give a characterization of left and right semiperfect corings in terms of duality, which generalizes the obtained for semiperfect coalgebras in [15, Theorem 3.5].

THEOREM 4.4. *Let A be a QF ring and $\mathfrak{C}$ an A–coring with ${}_A\mathfrak{C}$ and $\mathfrak{C}_A$ projective modules. Then $\mathfrak{C}$ is left and right semiperfect if and only if the functors* $\mathrm{Rat}^r_{\mathfrak{C}} \circ (-)^* : \mathcal{M}^{\mathfrak{C}} \leftrightarrows {}^{\mathfrak{C}}\mathcal{M} : \mathrm{Rat}^l_{\mathfrak{C}} \circ {}^*(-)$ *give a Colby-Fuller duality.*

Proof. Assume $\mathfrak{C}$ to be left and right semiperfect, so the subcategories $\mathcal{M}^{\mathfrak{C}}_f$ and ${}^{\mathfrak{C}}_f\mathcal{M}$ are linearly compact. By Theorem 3.2 and [11, Theorem 3], we get the Colby-Fuller duality. Conversely, apply [11, Theorem 3] to see that the subcategories of linearly compact objects $\mathcal{M}^{\mathfrak{C}}_f$ and ${}^{\mathfrak{C}}_f\mathcal{M}$ are linearly compact subcategories and then apply Theorem 4.2. □

Let us now develop the Morita duality theory for corings which extends the given in [16] in the coalgebra case. We will work over two ground rings A and B. We will use the same notation for the duals of modules over A or B. Which dual is acting each time will be clarified by the context.

THEOREM 4.5. *Let $\mathfrak{C}$ and $\mathfrak{D}$ be corings over QF rings A and B, respectively, such that ${}_A\mathfrak{C}$, ${}_B\mathfrak{D}$, $\mathfrak{C}_A$ and $\mathfrak{D}_B$ are projective modules. Assume that there exists a Colby-Fuller duality $H : \mathcal{M}^{\mathfrak{C}} \leftrightarrows {}^{\mathfrak{D}}\mathcal{M} : H'$. Then*

(a) *$\mathfrak{C}$, $\mathfrak{D}$ are left and right semiperfect corings.*
(b) *There exists a Colby-Fuller duality between ${}^{\mathfrak{C}}\mathcal{M}$ and $\mathcal{M}^{\mathfrak{D}}$.*

Proof. (*a*) By [16, Lemma 1.4] $\mathcal{M}^{\mathfrak{C}}_f$ and ${}^{\mathfrak{D}}_f\mathcal{M}$ are reflexive subcategories of $\mathcal{M}^{\mathfrak{C}}$ and ${}^{\mathfrak{D}}\mathcal{M}$, respectively; hence [11, Lemma 2, Theorem 3] imply that $\mathcal{M}^{\mathfrak{C}}_f$ and ${}^{\mathfrak{D}}_f\mathcal{M}$ are linearly compact subcategories. Therefore, by Theorem 4.2, $\mathfrak{C}$ is right semiperfect and $\mathfrak{D}$ is left semiperfect. Let us show that $\mathfrak{D}$ is right semiperfect; for this consider an arbitrary simple left $\mathfrak{D}$-comodule ${}_{\mathfrak{D}}S$. So $H'(S) \in \mathcal{M}^{\mathfrak{C}}_f$ is simple, hence $H'(S)^* \in {}^{\mathfrak{C}}_f\mathcal{M}$ is simple too. Since $\mathfrak{C}$ is a right semiperfect, ${}_AE(H'(S)^*)$ is finitely generated. Using Proposition 3.3, ${}^*E(H'(S)^*)_{\mathfrak{C}}$ is a finitely generated projective comodule, and ${}^*E(H'(S)^*) \twoheadrightarrow {}^*(H'(S)^*) \cong H'(S)$. If we apply the functor H to this last sequence, we get

$$S \cong HH'(S) \hookrightarrow H({}^*E(H'(S)^*)). \tag{8}$$

By [17, Proposición 3.3.10] or [16, Lemma 1.10], $H({}^*E(H'(S)^*))$ is a finitely generated injective left $\mathfrak{D}$-comodule. So ${}_AE(S)$ is a finitely generated module, by (8). This implies, by Theorem 3.5, that $\mathfrak{D}$ is right semiperfect. Analogously, $\mathfrak{C}$ is shown to be left semiperfect.

(*b*) It follows from part (*a*) and Theorem 4.4 that the following are Colby-Fuller dualities

$$F = \mathrm{Rat}^r_{\mathfrak{C}} \circ (-)^* : \mathcal{M}^{\mathfrak{C}} \leftrightarrows {}^{\mathfrak{C}}\mathcal{M} : \mathrm{Rat}^l_{\mathfrak{C}} \circ {}^*(-) = F',$$
$$G = \mathrm{Rat}^r_{\mathfrak{D}} \circ (-)^* : \mathcal{M}^{\mathfrak{D}} \leftrightarrows {}^{\mathfrak{D}}\mathcal{M} : \mathrm{Rat}^l_{\mathfrak{D}} \circ {}^*(-) = G'.$$

So, by [16, Lemma 1.4],

$$L = G'_f \circ H_f \circ F'_f : {}^{\mathfrak{C}}_f\mathcal{M} \leftrightarrows \mathcal{M}^{\mathfrak{D}}_f : L' = F_f \circ H'_f \circ G_f$$

is a duality, where $(-)_f$ denotes the restriction of a functor $(-)$ to the full subcategory of finitely generated comodules. Since ${}^{\mathfrak{C}}_f\mathcal{M}$ and $\mathcal{M}^{\mathfrak{D}}_f$ are finitely closed generating, linearly compact full subcategories of ${}^{\mathfrak{C}}\mathcal{M}$ and $\mathcal{M}^{\mathfrak{D}}$, respectively, the assertion (*b*) is deduced by [11, Theorem 5]. □

Let $\mathfrak{C}$ and $\mathfrak{D}$ be as in Theorem 4.5. If $L : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}^{\mathfrak{D}}$ is an equivalence of categories, then $L_f : \mathcal{M}^{\mathfrak{C}}_f \to \mathcal{M}^{\mathfrak{D}}_f$ is also an equivalence of categories. Moreover, the assignment $L \mapsto L_f$ defines a bijective correspondence (up to natural isomorphisms) between equivalences $\mathcal{M}^{\mathfrak{C}} \sim \mathcal{M}^{\mathfrak{D}}$ and equivalences $\mathcal{M}^{\mathfrak{C}}_f \sim \mathcal{M}^{\mathfrak{D}}_f$, see [16, Proposition 1.1]. On the other hand, if $\mathfrak{C}$ is right semiperfect and $\mathfrak{D}$ is a left semiperfect, then the assignment

$$H : \mathcal{M}^{\mathfrak{C}} \leftrightarrows {}^{\mathfrak{D}}\mathcal{M} : H' \longmapsto H_f : \mathcal{M}^{\mathfrak{C}}_f \leftrightarrows {}^{\mathfrak{D}}_f\mathcal{M} : H'_f$$

is also a bijective correspondence (up to natural isomorphism) between Colby-Fuller dualities and dualities, because each duality between $\mathcal{M}^{\mathfrak{C}}_f$ and ${}^{\mathfrak{D}}_f\mathcal{M}$ is uniquely extended to a Colby-Fuller duality, see [12, Theorem 2.1].

THEOREM 4.6. *Let $\mathfrak{C}$ and $\mathfrak{D}$ be corings over Quasi-Frobenius rings A and B, respectively, such that ${}_A\mathfrak{C}$, ${}_B\mathfrak{D}$, $\mathfrak{C}_A$, and $\mathfrak{D}_B$ are projective modules. Assume that $\mathfrak{C}$ is right semiperfect, and that $\mathfrak{D}$ is left semiperfect. There is a bijective correspondence (up to natural isomorphism) between equivalences $\mathcal{M}^{\mathfrak{C}} \sim \mathcal{M}^{\mathfrak{D}}$ and Colby-Fuller dualities $H : \mathcal{M}^{\mathfrak{C}} \leftrightarrows {}^{\mathfrak{D}}\mathcal{M} : H'$.*

Proof. Let $L : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}^{\mathfrak{D}}$ be an equivalence of categories, and let $L_f : \mathcal{M}^{\mathfrak{C}}_f \to \mathcal{M}^{\mathfrak{D}}_f$ the induced equivalence. Then $(-)^* \circ L_f : \mathcal{M}^{\mathfrak{C}}_f \to {}^{\mathfrak{D}}_f\mathcal{M}$ is a duality, which can be uniquely extended, by [11, Theorem 5] or [12, Theorem 2.1], to a Colby-Fuller duality between $\mathcal{M}^{\mathfrak{C}}$ and ${}^{\mathfrak{D}}\mathcal{M}$, as $\mathcal{M}^{\mathfrak{C}}_f$ and ${}^{\mathfrak{D}}_f\mathcal{M}$ are linearly compact subcategories by Theorem 4.2. Conversely, given a Colby-Fuller duality $H : \mathcal{M}^{\mathfrak{C}} \leftrightarrows {}^{\mathfrak{D}}\mathcal{M} : H'$, then ${}^*(-) \circ H_f : \mathcal{M}^{\mathfrak{C}}_f \to \mathcal{M}^{\mathfrak{D}}_f$ is an equivalence of categories which extends uniquely to an equivalence of categories between $\mathcal{M}^{\mathfrak{C}}$ and $\mathcal{M}^{\mathfrak{D}}$. □

It seems to be unknown for general corings $\mathfrak{C}$ and $\mathfrak{D}$ whether an equivalence of categories $\mathcal{M}^{\mathfrak{C}} \sim \mathcal{M}^{\mathfrak{D}}$ does imply an equivalence ${}^{\mathfrak{C}}\mathcal{M} \sim {}^{\mathfrak{D}}\mathcal{M}$. Up to our knowledge, this is far from being a trivial problem. Our next result gives a positive answer in a (restrictive) particular case.

THEOREM 4.7. *Let $\mathfrak{C}$ and $\mathfrak{D}$ be corings over Quasi-Frobenius rings A and B, respectively, such that ${}_A\mathfrak{C}$, ${}_B\mathfrak{D}$, $\mathfrak{C}_A$, and $\mathfrak{D}_B$ are projective modules. The following statements are equivalent.*

(i) *There is a Colby-Fuller duality $\mathcal{M}^{\mathfrak{C}} \leftrightarrows {}^{\mathfrak{D}}\mathcal{M}$;*
(ii) *there is an equivalence $\mathcal{M}^{\mathfrak{C}} \sim \mathcal{M}^{\mathfrak{D}}$, $\mathfrak{C}$ is right semiperfect, and $\mathfrak{D}$ is left semiperfect;*
(iii) *there is an equivalence ${}^{\mathfrak{C}}\mathcal{M} \sim {}^{\mathfrak{D}}\mathcal{M}$, $\mathfrak{C}$ is left semiperfect and $\mathfrak{D}$ is right semiperfect;*
(iv) *there is a Colby-Fuller duality ${}^{\mathfrak{C}}\mathcal{M} \leftrightarrows \mathcal{M}^{\mathfrak{D}}$.*

Proof. It follows easily from Theorems 4.5 and 4.6. □

To close this section we give a characterization of Colby-Fuller dualities between comodules by means of the existence of a quasi–finite injective cogenerator comodule; such characterization was given in [16, Corollary 1.8] for a coalgebras over a fixed base field. It is convenient, first, to recall the notion of quasi–finite comodule.

Let $\mathfrak{C}$ (resp. $\mathfrak{D}$) be a coring over A (resp. over B). Here, A and B are not assumed to be QF-rings. Let N be an $A-B$–bimodule with a right $\mathfrak{D}$–structure map $\rho_N : N \to N \otimes_B \mathfrak{D}$ which is assumed to be left A–linear. Assume that $N_{\mathfrak{D}}$ is quasi-finite, that is, the functor $- \otimes_A N : \mathcal{M}_A \to \mathcal{M}^{\mathfrak{D}}$ has a left adjoint $F : \mathcal{M}^{\mathfrak{D}} \to \mathcal{M}_A$, see [13, Section 4]. This functor is called the *cohom* functor by analogy with the case of coalgebras over fields (see [24]); notation $F = \mathrm{h}_{\mathfrak{D}}(N, -)$. Let $\eta_{-,-} : \mathrm{Hom}_{\mathfrak{D}}(-, - \otimes_A N) \to \mathrm{Hom}_A(F(-), -)$ denote the natural isomorphism of the adjunction, and $\theta : 1_{\mathcal{M}^{\mathfrak{D}}} \to F(-) \otimes_A N$ the unity of the adjunction. The canonical map $A_A \to \mathrm{Hom}_{\mathfrak{D}}(N, N) \to \mathrm{Hom}_A(F(N), F(N))$ gives a structure of left A–module on $F(N)$ such that $F(N)$ becomes an A–bimodule, which is endowed with a structure of A–coring as follows. Define a comultiplication $\Delta : F(N) \to F(N) \otimes_A F(N)$ by $\Delta = \eta_{N, F(N) \otimes_A F(N)}((F(N) \otimes_A \theta_N)\theta_N)$, that is, Δ is determined

by the condition $(F(N) \otimes_A \theta_N)\theta_N = (\Delta \otimes_A N)\theta_N$. The counit is given by $\epsilon = \eta_{N,A}(\iota)$, where $\iota : N \to A \otimes_A N$ is the canonical isomorphism. This A–coring will be denoted by $e_{\mathfrak{D}}(N)$.
Now, assume that N is a $\mathfrak{C} - \mathfrak{D}$–bicomodule and that ${}_B\mathfrak{D}$ is a flat module. By [13, Proposition 4.2], $F = \mathrm{h}_{\mathfrak{D}}(N, -)$ factors through the category $\mathcal{M}^{\mathfrak{C}}$, and $\mathrm{h}_{\mathfrak{D}}(N, -) : \mathcal{M}^{\mathfrak{D}} \to \mathcal{M}^{\mathfrak{C}}$ becomes a left adjoint to the cotensor product functor $-\square_{\mathfrak{C}} N : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}^{\mathfrak{D}}$ with unity $\theta : 1_{\mathcal{M}^{\mathfrak{D}}} \to F(-)\square_{\mathfrak{C}} N$ and counity $\chi : F(-\square_{\mathfrak{C}} N) \to 1_{\mathcal{M}^{\mathfrak{C}}}$.

PROPOSITION 4.8. [7] *Let N be a $\mathfrak{C} - \mathfrak{D}$-bicomodule. Assume that N is quasi-finite as a right $\mathfrak{D}$-comodule, and that ${}_B\mathfrak{D}$ is flat. The map $f : e_{\mathfrak{D}}(N) \to \mathfrak{C}$ defined by $f = \chi_{\mathfrak{C}} \circ \mathrm{h}_{\mathfrak{D}}(N, \lambda_N)$, where $\lambda_N : N \to \mathfrak{C} \otimes_A N$ is the left comodule structure map, is a homomorphism of A–corings.*

THEOREM 4.9. *Let $\mathfrak{C}$ and $\mathfrak{D}$ be corings over Quasi-Frobenius rings A and B, respectively, such that ${}_A\mathfrak{C}$, ${}_B\mathfrak{D}$, $\mathfrak{C}_A$, and $\mathfrak{D}_B$ are projective modules. If $H : \mathcal{M}^{\mathfrak{C}} \leftrightarrows {}^{\mathfrak{D}}\mathcal{M} : H'$ is a Colby-Fuller duality, then there exists a $\mathfrak{C} - \mathfrak{D}$-bicomodule P such that $P_{\mathfrak{D}}$ is a quasi-finite injective cogenerator of $\mathcal{M}^{\mathfrak{D}}$ with an isomorphism of A–corings $e_{\mathfrak{D}}(P) \cong \mathfrak{C}$, and a natural isomorphism*

$$H(M) \cong \mathrm{Rat}^r_{\mathfrak{D}}\left((M\square_{\mathfrak{C}} P)^*\right)$$

for every right $\mathfrak{C}$-comodule M.

Proof. Let $F : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}^{\mathfrak{D}}$ be the equivalence of categories corresponding by Theorem 4.6 to the given Colby–Fuller duality $H : \mathcal{M}^{\mathfrak{C}} \to {}^{\mathfrak{D}}\mathcal{M}$. Given $M \in \mathcal{M}^{\mathfrak{C}}$, write $M = \varinjlim M_i$, where the M_i's are finitely generated right $\mathfrak{C}$-comodules. Then

$$\begin{aligned} H(M) &= \mathrm{Rat}^r_{\mathfrak{D}}\left(\varprojlim (F_f(M_i))^*\right) \\ &\cong \mathrm{Rat}^r_{\mathfrak{D}}\left((\varinjlim F_f(M_i))^*\right) \\ &\cong \mathrm{Rat}^r_{\mathfrak{D}}\left((F(M))^*\right). \end{aligned}$$

By [13, Theorem 3.5] there is a $\mathfrak{C} - \mathfrak{D}$-bicomodule $P = F(\mathfrak{C})$ such that $F \cong -\square_{\mathfrak{C}} F(\mathfrak{C})$. Using [13, Proposition 4.2.(2)], we deduce that $P_{\mathfrak{D}}$ is a quasi-finite right comodule which is clearly an injective cogenerator. Now, $e_{\mathfrak{D}}(P) = F(\mathfrak{C})\square_{\mathfrak{D}} G(\mathfrak{D}) \cong GF(\mathfrak{C}) \cong \mathfrak{C}$, where G is the inverse functor of F; which is an isomorphism of A–corings, by Proposition 4.8. □

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